# GENERALIZED PROBLEM OF HEAT CONDUCTION FOR A HALF SPACE HEATED BY A MOVING POINT HEAT SOURCE

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Solutions of generalized heat-conduction problems are obtained for a heat-sensitive half space heated by a moving heat source.

In the movement of an electrical arc in the working chamber of a plasmatron [1, 2], the heat-source transfer rate can turn out to be comparable with the rate of heat propagation. The classical model may therefore prove to be insufficient for describing the temperature field. We consider, in connection with this, a generalized model, assuming that the thermophysical characteristics depend on the temperature.

If from the generalized heat-conduction law [3]

$$\mathbf{q} = -\lambda(t)\,\mathrm{grad}\,t - \tau_r \frac{d\mathbf{q}}{d\tau} \tag{1}$$

and the heat-balance equations

$$-\mathrm{div}\,\mathbf{q} = c(t)\frac{dt}{d\tau} - w \tag{2}$$

we eliminate the vector q, we obtain, for a constant convective transfer rate $v_k$, the following heat-conduction equation:

$$\mathrm{div}\,[\lambda(t)\,\mathrm{grad}\,t] = \left(1 + \tau_r \frac{d}{d\tau}\right)\left[c(t)\frac{dt}{d\tau} - w\right], \tag{3}$$

where the total derivative of w with respect to the time has the form [3]

$$\frac{dw}{d\tau} = \frac{\partial w}{\partial \tau} + (\mathbf{v}_\text{к}\,\mathrm{grad}\,w),$$

and the operator $d^2/d\tau^2$ is of the form

$$\frac{d^2}{d\tau^2} = \frac{\partial^2}{\partial\tau^2} + 2(\mathbf{v}_\text{к}\,\mathrm{grad})\frac{\partial}{\partial\tau} + (\mathbf{v}_\text{к}\,\mathrm{grad})(\mathbf{v}_\text{к}\,\mathrm{grad}).$$

When $v_k = 0$ Eq. (3) reverts to a well-known form [4].

We consider an isotropic half space $z \geq 0$ whose thermophysical characteristics depend on the temperature. The half-space is heated by a point heat source moving with constant speed v a) over the surface z = 0+ along the Oy axis, b) perpendicular to it (along the Oz axis). Let the surface z = 0 be thermally insulated. The temperature and its derivatives with respect to the coordinates at infinity, as well as the initial temperature and heating rate are equal to zero. We then have, for the determination of the generalized three-dimensional nonstationary temperature field in the half space, the heat-conduction equation [4]

$$\mathrm{div}\,[\lambda(t)\,\mathrm{grad}\,t] = lc(t)\,\dot{t} - q\delta(x)\,lS_+(\tau)\begin{cases}\delta_+(z)\,\delta(y - v\tau),\\ \delta(y)\,\delta_+(z - v\tau),\end{cases} \tag{4}$$

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where

$$l = 1 + \tau_r \frac{\partial}{\partial \tau}, \quad \delta_+(\zeta) = \frac{dS_+(\zeta)}{d\zeta}, \quad \delta(\zeta) = \frac{dS(\zeta)}{d\zeta}, \quad \dot{t} = \frac{\partial t}{\partial \tau},$$

$$S_\pm(\zeta) = \begin{cases} 1, & \zeta > 0, \\ 0.5 \mp 0.5, & \zeta = 0, \\ 0, & \zeta < 0, \end{cases} \qquad S(\zeta) = \begin{cases} 1, & \zeta > 0, \\ 0{,}5, & \zeta = 0, \\ 0, & \zeta < 0, \end{cases}$$

$$-\infty \leqslant x \leqslant \infty, \quad -\infty \leqslant y \leqslant \infty, \quad 0 \leqslant z \leqslant \infty, \quad 0 \leqslant \tau \leqslant \infty.$$

The boundary conditions are written in the form $\left.\frac{\partial t}{\partial z}\right|_{z=0} = 0$ , (5)

$$\left\{t, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}, \frac{\partial t}{\partial z}\right\}_\infty = 0, \quad t|_{\tau=0} = \dot{t}|_{\tau=0} = 0. \tag{6}$$

Introducing the Kirchhoff variable [5]

$$\Theta = \frac{1}{\lambda_0} \int_0^t \lambda(t)\, dt \tag{7}$$

and taking into account the fact that for many metals (see [6]) a ≈ const, we obtain in place of Eqs. (4) and (6)

$$\Delta\Theta = \frac{l\dot{\Theta}}{a} - \frac{q}{\lambda_0} \delta(x) l S_+(\tau) \begin{cases} \delta_+(z)\,\delta(y - v\tau), \\ \delta(y)\,\delta_+(z - v\tau), \end{cases} \tag{8}$$

$$\left.\frac{\partial\Theta}{\partial z}\right|_{z=0} = 0, \tag{9}$$

$$\left\{\Theta, \frac{\partial\Theta}{\partial x}, \frac{\partial\Theta}{\partial y}, \frac{\partial\Theta}{\partial z}\right\}_\infty = 0, \quad \Theta|_{\tau=0} = 0, \quad \dot{\Theta}|_{\tau=0} = 0, \tag{10}$$

where

$$a \approx \frac{c(t)}{\lambda(t)} = \frac{c_0}{\lambda_0}.$$

To solve the problem we employ in the case a) Fourier integral transformations with respect to x and y and a Laplace transformation with respect to τ; in case b) cosine Fourier transformations with respect to z, Fourier with respect to x, and Laplace with respect to τ, taking into account the necessary conditions of (9), (10).

$$\frac{d^2\overline{\Theta}}{dz^2} - \gamma^2\overline{\Theta} = -\frac{q}{\lambda_0}\delta_+(z)\overline{\varphi}, \tag{11}$$

$$\frac{d^2\hat{\Theta}}{dy^2} - \gamma_0^2\hat{\Theta} = -\frac{q}{\lambda_0}\delta(y)\hat{\psi}, \tag{12}$$

where

$$\overline{\Theta} = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} \Theta \exp(i(\xi x + \eta y) - s\tau)\, dx dy d\tau;$$

$$\hat{\Theta} = \frac{1}{\pi} \int_{-\infty}^{\infty}\int_0^{\infty}\int_0^{\infty} \Theta \exp(i\xi x - s\tau) \cos\zeta z dx dz d\tau;$$

$$\gamma=\sqrt{\xi^2+\eta^2+\frac{s}{a}+\frac{s^2}{c_q^2}},\quad \gamma_0=\sqrt{\xi^2+\zeta^2+\frac{s}{a}+\frac{s^2}{c_q^2}},$$

$$c_q=\sqrt{\frac{a}{\tau_r}};$$

$$\bar{\varphi}=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty} l\,[S_+(\tau)\,\delta\,(y-v\tau)]\exp\,(i\eta y-s\tau)\,dyd\tau;$$

$$\hat{\psi}=\frac{1}{\pi}\int\limits_{0}^{\infty}\int\limits_{0}^{\infty} l\,[S_+(\tau)\,\delta_+(z-v\tau)]\cos\zeta z\exp\,(-s\tau)\,dzd\tau.$$

The solutions of Eqs. (11) and (12), taking into account the transformed boundary conditions at the surface z = 0 and at infinity, may be written, respectively, as follows:

$$\bar{\Theta}=\frac{q}{2\lambda_0\gamma}\,\bar{\varphi}\,(\exp\,(-\gamma z)+\exp\,(-\gamma\,|z|_+)),\tag{13}$$

$$\hat{\Theta}=\frac{q}{4\lambda_0\gamma_0}\,\hat{\psi}\,(\exp\,(-\gamma_0\,|y|_+)+\exp\,(-\gamma_0\,|y|_-)),\tag{14}$$

where

$$|z|_\pm=z\,\mathrm{sign}_\pm\,z,\quad \mathrm{sign}_\pm z=2S_\pm(z)-1.$$

Since

$$\delta\,(y)=\frac{1}{2}\,[\delta_+(y)+\delta_-(y)],\quad \mathrm{sign}_\pm^2 y=1,\quad \mathrm{sign}_\mp^2 z=1,$$

it is not difficult to see that function (13) satisfies Eq. (11) and function (14) satisfies Eq. (12).

We now take the inverse transforms of functions (13) and (14), using the reference data and the convolution theorems for the Fourier transform, the Fourier cosine-transform, and the Laplace transform [7, 8]. As a result, we obtain expressions for the Kirchhoff variable for each of the cases:

$$\Theta=\frac{q}{2\lambda_0\pi^2}\,S_+(\tau-b)\left\{\int\limits_{b}^{\tau}\frac{\exp\left(-\frac{\tau_1}{2\tau_r}\right)}{\sqrt{\tau_1^2-b^2}}\int\limits_{0}^{\infty}\mathrm{ch}\,\frac{1}{2\tau_r}\times\right.$$
$$\times\sqrt{\left(1-4\eta^2\frac{a^2}{c_q^2}\right)(\tau_1^2-b^2)}\,[\cos\,(y_1+v\tau_1)\,\eta+\tau_r v\eta\sin\,(y_1+v\tau_1)\,\eta]\,d\eta d\tau_1+$$
$$\left.+\frac{\tau_r}{\sqrt{\tau^2-b^2}}\int\limits_{0}^{\infty}\mathrm{ch}\,\frac{1}{2\tau_r}\sqrt{\left(1-4\eta^2\frac{a^2}{c_q^2}\right)(\tau^2-b^2)}\cos\eta y d\eta\right\},\tag{15}$$

$$\Theta=\frac{q}{4\lambda_0\pi^2}\,S_+(\tau-c)\left\{\int\limits_{c}^{\tau}\frac{\exp\left(-\frac{\tau_1}{2\tau_r}\right)}{\sqrt{\tau_1^2-c^2}}\int\limits_{0}^{\infty}\mathrm{ch}\,\frac{1}{2\tau_r}\times\right.$$
$$\times\sqrt{\left(1-4\zeta^2\frac{a^2}{c_q^2}\right)(\tau_1^2-c^2)}\,[\cos\,(z_-+v\tau_1)\,\zeta+\tau_r v\zeta\sin\,(z_-+v\tau_1)]\,d\zeta d\tau_1+$$
$$+\frac{2\tau_r}{\sqrt{\tau^2-c^2}}\int\limits_{0}^{\infty}\mathrm{ch}\,\frac{1}{2\tau_r}\sqrt{\left(1-4\zeta^2\frac{a^2}{c_q^2}\right)(\tau^2-c^2)}\cos\zeta z d\zeta+$$

$$+\int_{c}^{\tau}\frac{\exp\left(-\frac{\tau_1}{2\tau_r}\right)}{\sqrt{\tau_1^2-c^2}}\int_{0}^{\infty}\operatorname{ch}\frac{1}{2\tau_r}\sqrt{\left(1-4\zeta^2\frac{a^2}{c_q^2}\right)(\tau_1^2-c^2)}\,[\cos(z_+-v\tau_1)\,\zeta-\tau_r v\zeta\sin(z_+-v\tau_1)]\,d\zeta d\tau_1\Bigg\}, \quad (16)$$

where

$$b=\frac{\sqrt{x^2+z^2}}{c_q},\quad y_1=y-v\tau,\quad c=\frac{\sqrt{x^2+y^2}}{c_q},\quad z_\pm=z\pm v\tau.$$

For a fixed heat source (v = 0) we have

$$\Theta=\frac{q}{2\pi^2\lambda_0}S_+(\tau-b)\left[\int_0^{\tau}\exp\left(-\frac{\tau_1}{2\tau_r}\right)\int_0^{\infty}\operatorname{ch}\frac{1}{2\tau_r}\times\right.$$

$$\times\sqrt{\left(1-4\eta^2\frac{a^2}{c_q^2}\right)(\tau_1^2-b^2)}\cos\eta y d\eta\frac{d\tau_1}{\sqrt{\tau_1^2-b^2}}+\frac{\tau_r}{\sqrt{\tau^2-b^2}}\times$$

$$\left.\times\exp\left(-\frac{\tau}{2\tau_r}\right)\int_0^{\infty}\operatorname{ch}\frac{1}{2\tau_r}\sqrt{\left(1-4\eta^2\frac{a^2}{c_q^2}\right)(\tau^2-b^2)}\cos\eta y d\eta\right]. \quad (17)$$

For the classical case $(\tau_r = 0,\ c_q \to \infty)$ it follows from Eqs. (15), (16) that

$$\Theta=\frac{q}{4\pi\lambda_0 R_1}\exp(-y_1\omega)\left[\exp(-R_1\omega)\operatorname{erfc}\left(\frac{R_1}{2\sqrt{\tau a}}-\omega\sqrt{\tau a}\right)+\right.$$

$$\left.+\exp(R_1\omega)\operatorname{erfc}\left(\frac{R_1}{2\sqrt{\tau a}}+\omega\sqrt{\tau a}\right)\right]=Q\Phi(y_1,\ R_1,\ \tau), \quad (18)$$

$$\Theta=\frac{Q}{2}[\Phi(z_-,\ R_-,\ \tau)+\Phi(-z_+,\ R_+,\ \tau)], \quad (19)$$

where

$$\omega=\frac{v}{2a},\quad R_1=\sqrt{x^2+y_1^2+z^2},\quad R_\pm=\sqrt{x^2+y^2+z_\pm^2},\quad \operatorname{erfc}\zeta=1-\operatorname{erf}\zeta.$$

Letting $\tau \to \infty$, we obtain from Eqs. (18), (19) the solutions

$$\Theta=2\frac{Q}{R_1}\exp(-(y_1+R_1)\omega), \quad (20)$$

$$\Theta=Q[\exp(-(z_1+R_-)\omega)R_-^{-1}+\exp(-(R_+-z_+)\omega)R_+^{-1}], \quad (21)$$

corresponding to a quasistationary thermal regime.

These solutions are given in [6, 9-11] for the case in which the thermophysical characteristics do not depend on the temperature. For the sources moving along the surface z = 0+, a solution was obtained by another method starting from the differential equation [9]

$$\Delta_1 t+2\omega\frac{\partial t}{\partial y_1}=0, \quad (22)$$

where

$$\Delta_1=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial z^2}.$$

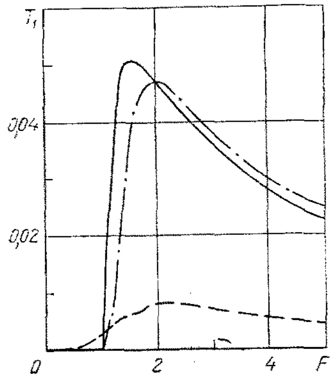


Fig. 1

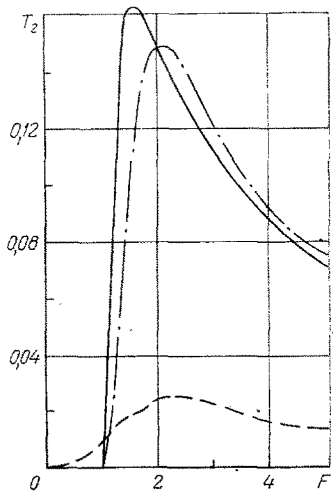


Fig. 2

Fig. 1. Variation of temperature $T_1$ = T/8 as a function of the parameter F for $\beta$ = 0.25.

Fig. 2. Variation of temperature $T_2$ = 3T/8 as a function of the parameter F for $\beta$ = 0.75.

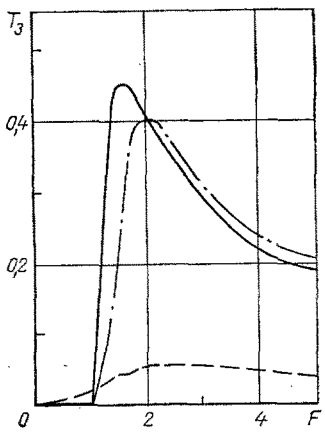


Fig. 3

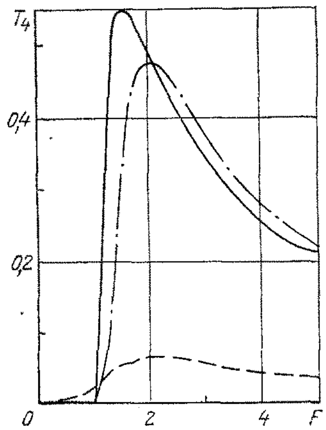


Fig. 4

Fig. 3. Variation of temprature $T_3$ = 7T/8 as a function of the parameter F for $\beta$ = 1.75.

Fig. 4. Variation of temperature $T_4$ as a function of the parameter F for $\beta$ = 2.

In our case, in accordance with Eq. (4), the differential equation has the form

$$\Delta_1 t + 2\omega \frac{\partial t}{\partial y_1} = -\frac{q}{\lambda}\,\delta(x)\,\delta(y_1)\,\delta_+(z), \tag{23}$$

i.e., the action of the heat source is accounted for in the differential equation, whereas in the aforementioned papers it is taken into account in the boundary condition. Here $\lambda$ = const.

In [9], in determining the thermoelastic displacement potential, $\partial t/\partial y_1$ is determined from Eq. (22). But if we determine $\partial t/\partial y_1$ from Eq. (23), we arrive at the equation

$$\Delta_1 \left( \frac{\partial \Phi}{\partial y_1} + \frac{1+\nu}{1-\nu}\,\frac{\alpha_t}{2\omega}\,t \right) = -\frac{1+\nu}{1-\nu}\,\frac{q}{2\lambda\omega}\,\alpha_t \delta(x)\,\delta(y_1)\,\delta_+(z), \tag{24}$$

whose solution, as is well known [12], has the form

$$\frac{\partial \Phi}{\partial y_1}+\frac{1+\nu}{1-\nu}\,\frac{\alpha_t}{2\omega}\,t=-\frac{q}{8\pi\lambda\omega}\,\frac{1+\nu}{1-\nu}\,\frac{\alpha_t}{R_1}. \tag{25}$$

Consequently, we may write the thermoelastic displacement potential as follows:

$$\Phi=-\frac{1+\nu}{1-\nu}\,\frac{\alpha_t}{2\omega}\int\left(t+\frac{q}{4\pi\lambda R_1}\right)dy_1+\Phi_0. \tag{26}$$

An analogous situation appears in determining the thermoelastic displacement potential in an finite plate heated by a moving heat source. In determining the thermoelastic displacement potential in [9] no account was taken of the right side $q/2\lambda\delta\ \delta(x, y_1)$ of the differential heat-conduction equation for determining the quasistationary temperature field; as a result of this, certain terms are missing in the expressions for the thermal stresses. This was first noted in [13], It was remarked upon later by V. A. Vinokurov [14].

For aluminum, for example, $\tau_r = 10^{-11}$ sec from [15]; hence, $c_q = 3\cdot 10^5$ cm/sec. The thermal conductivity coefficient for aluminum varies linearly with the temperature in accordance with the law [5]

$$\lambda(t)=\lambda_0(1-kt). \tag{27}$$

Consequently, the temperature field t, in terms of the Kirchhoff variable θ, is expressed as follows:

$$t=\frac{1}{k}\left(1-\sqrt{1-2k\Theta}\right). \tag{28}$$

Knowing the variable θ and the magnitude of the heat propagation rate $c_q$, we can calculate from relation (28) the generalized temperature field in a half space heated by a moving heat source.

We consider now the case in which the heat source displacement speed is equal to the speed of heat propagation. It follows from Eq. (12) with $c_q$ = v that

$$\Theta=S_+(\tau)\left\{2QS(\tau_1)\exp\left(-\frac{x^2+z^2}{4a\tau_1}\right)\frac{1}{v\tau_1}\left[1+\frac{a}{\tau_1 v^2}\left(\frac{x^2+z^2}{4a\tau_1}-1\right)\right]+\frac{2a^2q}{\lambda_0 v^3}\delta(\tau_1)\,\delta(x)\,\delta_+(z)\right\}, \tag{29}$$

where $\tau_1 = \tau - y/v$.

If in Eq. (26) we neglect inertia of the heat source [16], we arrive at the expression

$$\Theta=\frac{2Q}{v\tau_1}S_+(\tau)\,S(\tau_1)\exp\left(-\frac{x^2+z^2}{4a\tau_1}\right). \tag{30}$$

From the Eqs. (28)-(30) we have calculated the temperature field in a halfspace for $\xi=xv/2a=\sqrt{2}$, $\eta=yv/2a=2$, $\zeta=zv/2a=\sqrt{2}$ after transforming these equations to the dimensionless form:

$$T=\frac{2}{\beta}\left(1-\sqrt{1-\beta\vartheta}\right), \tag{31}$$

$$\vartheta=\frac{S_+(F)}{F-1}\exp\left(-\frac{1}{F-1}\right)S(F-1)\left[1+\frac{1}{4(F-1)}\left(\frac{1}{F-1}-1\right)\right], \tag{32}$$

$$\vartheta=\frac{S_+(F)}{F-1}\exp\left(-\frac{1}{F-1}\right)S(F-1), \tag{33}$$

where $\beta=8k\omega Q$; $T=t/4Q\omega$; $\vartheta=\Theta/4Q\omega$; $F=(v^2\tau)/4a$ .

The corresponding solution of the classical problem, according to (15), may be written as follows:

$$\vartheta=\frac{1}{8\sqrt{1+(1-F)^2}}\exp\left[-2(1-F)\right]\left[\exp\left[-2\sqrt{1+(1-F)^2}\right]\right.\times \tag{34}$$

$$\times \operatorname{erfc}\left(\frac{\sqrt{1+(1-F)^2}}{\sqrt{F}}-\sqrt{F}\right)+\exp\left(2\sqrt{1+(1-F)^2}\right)\operatorname{erfc}\left(\frac{\sqrt{1+(1-F)^2}}{\sqrt{F}}+\sqrt{F}\right)\Bigg]. \quad (34)$$

The results of our calculations are shown in Figs. 1-4. In these figures the continuous, the dash-dot, and the dash curves correspond to the solutions (31), (32); (31), (33); and (31), (34). These graphs show that the largest value of the temperature is attained with the complete generalized solution; moreover, the maximum value is reached considerably sooner than with the incomplete generalized (neglecting inertia of the heat source) and classical solution.

## NOTATION

$\lambda(t)$, thermal conductivity; $c(t)$, coefficient of volumetric heat capacity; $S_+(\zeta)$, $S(\zeta)$, asymmetric and symmetric unit functions; $\tau$, time; $t$, temperature field; $q$, heat source power; erf $\zeta$, probability integral; $\tau_r$, relaxation time of the heat flux; $c_q$, heat propagation rate; $a$, thermal diffusivity; $\lambda_0$, reference thermal conductivity; $c_0$, reference volumetric heat capacity; $s$, Laplace transformation parameter; $\xi$, $\eta$, Fourier transformation parameters along $x$, $y$; $\zeta$, Fourier cosine transformation parameter along $z$; $\nu$, Poisson coefficient; $\alpha_t$, temperature coefficient of linear expansion.

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